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The Matrix Nonlinear Schrödinger Equation in Dimension 2

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In this paper we study the existence of global solutions to the Cauchy problem for the matrix nonlinear Schrödinger equation (MNLS) in 2 space dimensions. A sharp condition for the global existence is obtained for this equation. This condition is in terms of an exact stationary solution of a semilinear elliptic equation. In the scalar case, the MNLS reduces to the well-known cubic nonlinear Schrödinger equation for which existence of solutions has been studied by many authors.

Key Words: matrix nonlinear Schrödinger equation; Cauchy problem; ground state solution.

1. INTRODUCTION

This paper deals with the existence of solutions of the following matrix nonlinear Schrödinger equation

$$B_t = i(\Delta B + 2BB^*B), \tag{1.1}$$

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where Δ is the Laplace operator on R^d, B is a map from $M \times [0, +\infty)$, where $M \subseteq R^d$, to the space $M_{n\times m}$ of complex $n \times m$ matrices, and $B^* = \overline{B}^t$ denotes the adjoint of B. When $n = m = 1$, (1.1) is the well-known (scalar) cubic nonlinear Schrödinger equation (for more general forms of the nonlinear Schrödinger equation, see [7]). In [1–3], Bourgain studied the Cauchy problem for the scalar equation under the periodic boundary condition, i.e., for $M = \overline{T}^d = R^d/Z^d$, with the following Cauchy data,

$$B(\cdot, 0) = B_0 \in H^k(M). \tag{1.2}$$

Fordy and Kulish [6] first studied the generation of nonlinear Schrödinger equations to matrix nonlinear Schrödinger equations for $M = R^1$ by using inverse scattering methods and established the correspondence between solutions of the matrix nonlinear Schrödinger equation on R^1 and Schrödinger flows from R^1 to complex Grassmannian manifolds in this case. Pang *et al*. [10] established the existence and uniqueness of a global solution for the matrix nonlinear equation with $M = \overline{T}^1$ and $M = R^1$.

The object of our study is the Cauchy problem (1.1)–(1.2), with $M = R^2$, of the matrix nonlinear equation. Our main result is as follows:

THEOREM 1. *Let* $M = R^2$, $B_0 \in H^l(R^2, M_{n\times m})$ $(l \geq 1)$. *The Cauchy problem* (1.1)–(1.2) *has a global solution* $B \in L_2^\infty((0, +\infty), H^\infty(R^2))$, *if*

$$\|B_0\|_{L^2(R^2)} < \|\psi\|_{L^2(R^2)}. \tag{1.3}$$

Here ψ *is a positive solution of the equation*

$$\Delta u - u + u^3 = 0 \qquad \text{in } R^2 \tag{1.4}$$

of minimal L^2*-norm* (*the ground state*). *Moreover*, *if* $l \geq 3$, *the solution is unique*.

Remark 1. The condition (1.3) is a sharp condition for the Cauchy problem (1.1)–(1.3) to have a global solution.

Let us note that in the work that follows, the basic tool is energy estimates, whereas similar to the scalar case, the Strichartz estimates for the matrix nonlinear Schrödinger equation

$$\begin{cases} B_t = i(\Delta B + 2BB^*B), & (t, x) \in R \times R^2, \\ B(\cdot, 0) = B_0, & x \in R^2, \end{cases} \tag{1.5}$$

provide the following local existence result [1]:

LEMMA 1. *Suppose* $B_0 \in H^k(R^2, M_{n\times m})$. *Then there exists a* $T^* = T^*(\|B_0\|_{H^k}) > 0$ *such that the Cauchy problem* (1.5) *has a solution*

$$B \in C^0\big([0, T^*], H^k\big(R^2, M_{n\times m}\big)\big).$$

Furthermore

$$B \in L^q\big(0, T^*, W^{k,r}(R^n, M_{n\times m})\big)$$

for every $T < T^*$, *where*

$$\frac{1}{q} = \frac{1}{2} - \frac{1}{r}, \qquad r \geq 2.$$

2. LOCAL EXISTENCE

We first note that the matrix nonlinear Schrödinger equation is an infinite dimensional Hamiltonian system [5]; indeed, it corresponds to the Hamiltonian functional

$$H(B) = \int_{R^2} \sum_i tr\big(B_{x_i} B^*_{x_i}\big) - tr(BB^*BB)\,dx \tag{2.1}$$

defined on $H^1(R^2, M_{n\times m})$ with the symplectic form

$$\omega(B_1, B_2) = \int_{R^2} \langle -iB_1, B_2 \rangle\,dx. \tag{2.2}$$

Here $\langle\cdot,\cdot\rangle$ is the Hermitian inner product on $M_{n\times m}$ given by

$$\langle B_1, B_2 \rangle = Re\, tr(B_1 B_2^*). \tag{2.3}$$

Thus, conservation laws are provided by the L^2-norm and the Hamiltonian along the solution, namely, if $B(\cdot, t) \in H^1(R^2, M_{n\times m})$ is a solution to the matrix nonlinear Schrödinger equation, then we have

LEMMA 2.1. *As long as* $B(\cdot, t)$ *remains in* $H^1(R^2)$,

$$\mathcal{N}(B) = \int_{R^2} |B(x,t)|^2\,dx \tag{2.4}$$

and

$$H(B) = \int_{R^2} \sum_i \langle B_{x_i}, B_{x_i} \rangle - \langle BB^*, BB^* \rangle\,dx \tag{2.5}$$

are constants in time.

To solve the Cauchy problem (1.1)–(1.2), as in [10], our strategy is to study an approximate Cauchy problem of Landau–Lifshitz type:

$$\begin{cases} B_t = \varepsilon \Delta B + i(\Delta B + 2BB^*B) \\ B(\cdot, 0) = B_0 \end{cases} \tag{2.6}$$

and then let $\varepsilon \to 0^+$.

In view of its uniform parabolicity, for each $\varepsilon > 0$, (2.6) has a unique (local) classical solution B^ε on $R^2 \times [0, T_\varepsilon)$.

To establish a uniform lower bound for T_ε, one needs the following lemma which will be proved.

LEMMA 2.2. *Let $\varepsilon \in (0,1]$, and $B = B^\varepsilon$ be a local classical solution of* (2.6). *There exists a constant $T > 0$, which is independent of ε but depends on B_0, such that the following estimate holds for all $t \in (0, T]$:*

$$\|B\|_{H^l(R^2, M_{n\times m})} \leq C(l, B_0). \tag{2.7}$$

Here $C(l, B_0)$ denotes a constant depending only on the Sobolev constant of R^2, the parameter l, and Cauchy data B_0; in particular, C is independent of ε.

Proof. First, we verify the result for $l = 0$ by direct computation:

$$\begin{aligned} \frac{d}{dt}\int_{R^2}|B|^2\,dx &= 2\int_{R^2}\langle B_t, B\rangle\,dx \\ &= 2\int_{R^2}\langle(\varepsilon + i)\Delta B + 2iBB^*B, B\rangle\,dx \\ &= -2\varepsilon\int_{R^2}|\nabla B|^2 dx \leq 0; \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{R^2}|\nabla B|^2 dx &= \int_{R^2}\langle\nabla B_t, \nabla B\rangle\,dx \\ &= \int_{R^2}\sum_j\langle\varepsilon(i\Delta B + 2iBB^*B)_{x_j}, B_{x_j}\rangle\,dx \\ &= -\varepsilon\int_{R^2}|\Delta B|^2 + \sum_j\langle 2i(BB^*B)_{x_j}, B_{x_j}\rangle\,dx \\ &\leq C\int_{R^2}|\nabla B|^2|B|^2 dx. \end{aligned} \tag{2.9}$$

By the Sobolev imbedding theorem, we have

$$\|B\|_{L^\infty(R^2)} \leq C\|B\|_{H^2(R^2)}, \tag{2.10}$$

where C is the Sobolev constant of R^2. Thus, by (2.9), we have

$$\frac{d}{dt}\int_{R^2}|\nabla B|^2 dx \leq C\|B\|^4_{H^2(R^2)}. \tag{2.11}$$

Similarly,

$$\frac{1}{2}\frac{d}{dt}\int_{R^2}|\nabla^2 B|^2 dx = \int_{R^2}\Sigma_{i,j}\langle (B_t)_{x_i x_j}, B_{x_i x_j}\rangle\, dx$$

$$= -\varepsilon\int_{R^2}|\nabla^3 B|^2 + 2\langle i(BB^*B)_{x_i x_j}, B_{x_i x_j}\rangle\, dx$$

$$\leq C\int_{R^2}|\nabla^2 B|^2|B|^2 + |\nabla^2 B|\,|\nabla B|^2\,|B|\,dx.$$

By (2.10), we have

$$\int_{R^2}|\nabla^2 B|^2|B|^2 dx \leq \|B\|_{L^\infty}^2\int_{R^2}|\nabla^2 B|^2\,dx$$

$$\leq C\|B\|_{H^2(R^2)}^4,$$

$$\int_{R^2}|\nabla^2 B|^2\cdot|\nabla B|\cdot|B|\,dx \leq \|B\|_{L^\infty}\int_{R^2}|\nabla^2 B|\,|\nabla B|^2\,dx$$

$$\leq \|B\|_{L^\infty}\left(\int_{R^2}|\nabla^2 B|^2\,dx\right)^{1/2}\left(\int_{R^2}|\nabla B|^4 dx\right)^{1/2}.$$

By the Sobolev imbedding theorem,

$$\|\nabla B\|_{L^r(R^2)} \leq C\|B\|_{H^2(R^2)}, \qquad \text{for any } r > 2. \tag{2.12}$$

Hence

$$\int_{R^2}|\nabla^2 B|\cdot|\nabla B|^2\cdot|B|dx \leq C\|B\|_{H^2(R^2)}^4.$$

So we obtain

$$\frac{d}{dt}\int_{R^2}|\nabla^2 B|^2 dx \leq C\|B\|_{H^2(R^2)}^4. \tag{2.13}$$

Thus, summing (2.8), (2.11), (2.13) one gets

$$\frac{d}{dt}\|B\|_{H^2(R^2)}^4 \leq C\|B\|_{H^2(R^2)}^4, \tag{2.14}$$

where C is a constant independent of ε.

This ordinary differential inequality implies that for any constant $K > \|B_0\|_{H^2(R^2)}^2$, we can find $T^* = T^*(R^2)$ such that

$$\|B\|_{H^2(R^2)}^2 \leq 2K \tag{2.15}$$

for any $t \in [0, T^*]$.

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\int_{R^2}|\nabla^3 B|^2 dx = \int_{R^2}\Sigma\langle (B_t)_{x_i x_j x_k}, B_{x_i x_j x_k}\rangle\, dx$$
$$\le C\int_{R^2}|\nabla^3 B|^2|B|^2 + |\nabla^3 B|\,|\nabla B|\,|B| + |\nabla^3 B|\,|\nabla B|^3\, dx$$
$$:= C(A_1 + A_2 + A_3). \tag{2.16}$$

By the Gagliardo–Nirenberg inequality, we have

$$\left(\int_{R^2}|\nabla^2 B|^4 dx\right)^{1/4} \le \left(\int_{R^2}|\nabla^3 B|^2 dx\right)^{1/2}\left(\int_{R^2}|\nabla^2 B|^2 dx\right)^{1/2}$$
$$\le C\left(R^2, K, B_0\right)\left(\int_{R^2}|\nabla^3 B|^2 dx\right)^{1/2},$$
$$\left(\int_{R^2}|\nabla B|^4 dx\right)^{1/4} \le \left(\int_{R^2}|\nabla^2 B|^2 dx\right)^{1/2}\left(\int_{R^2}|\nabla B|^2 dx\right)^{1/2}$$
$$\le C\left(R^2, K, B_0\right).$$

Thus,

$$A_2 \le \|B\|_{L^\infty}\int_{R^2}|\nabla^3 B|\,|\nabla^2 B|\,|\nabla B| dx$$
$$\le \|B\|_{L^\infty}\left(\int_{R^2}|\nabla^3 B|^2 dx\right)^{1/2}\left(\int_{R^2}|\nabla^2 B|^4 dx\right)^{1/4}\left(\int_{R^2}|\nabla B|^4 dx\right)^{1/4}$$
$$\le C\left(R^2, K, B_0\right)\int_{R^2}|\nabla^3 B|^2 dx. \tag{2.17}$$

Similarly, we obtain

$$A_3 \le C\left(R^2, K, B_0\right)\int_{R^2}|\nabla^3 B|^2 dx. \tag{2.18}$$

Thus, it follows that for $t \in [0, T^*(K)]$,

$$\frac{d}{dt}\int_{R^2}|\nabla^3 B|^2 dx \le C\left(R^2, K, B_0\right)\int_{R^2}|\nabla^3 B|^2 dx.$$

Consequently, we obtain

$$\int_{R^2}|\nabla^3 B|^2 dx \le C\left(R^2, K, B_0\right), \qquad 0 \le t \le T^*(K). \tag{2.19}$$

It is now apparent, proceeding by induction, that

$$\int_{R^2} |\nabla^l B|^2 dx \le C\big(R^2, l, K, B_0\big), \qquad 0 \le t \le T^*(K). \tag{2.20}$$

Let $T = T^*(K)$ and the proof of Lemma 2.2 is complete.

From Lemma 2.2, we can complete the proof of local solvability. Using the uniform (with respect to ε) bounds on the $H^l(R^2, M_{n\times m})$-norm of B_ε for all l and $t \in (0, T^*]$, by a standard argument, we can infer the existence of a sequence $\{B_\varepsilon\}$ such that the limit B is a solution to (1.1)–(1.2) on $R^2 \times [0, T^*]$.

LEMMA 2.3. *Let B be a $L^\infty((0, +\infty), C^2(R^2))$-solution to the Cauchy problem* (1.1)–(1.2) *with Cauchy data B_0. Then B is unique.*

Proof. Let B^1, B^2: $R^2 \times [0, T] \to M_{n\times m}$ be two solutions to the Cauchy problem (1.1)–(1.2) with the same Cauchy data B_0. Let $\tilde{B} = B^1 - B^2$. Then

$$\tilde{B}_t = i\Big\{\Delta \tilde{B} + 2\big[B^1 B^{1*}\tilde{B} + (B^1 B^{1*} - B^2 B^{2*}) B^2\big]\Big\}.$$

By integration by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_{R^2} |\tilde{B}|^2 dx = \int_{R^2} \langle \tilde{B}, \tilde{B}_t \rangle\, dx \le C\int_{R^2} |\tilde{B}|^2 dx,$$

where C depends only on the $L^\infty((0, +\infty), C^2(R^2))$-norm of B^1 and B^2. As $\tilde{B}(x, 0) = 0$, by the Gronwall inequality,

$$\int_{R^2} |\tilde{B}|^2 dx = 0 \qquad \text{for all } t \in [0, T],$$

and the proof of Lemma 2.3 is complete.

Note that if $B \in L^\infty((0, +\infty), H^l(R^2))$, $l \ge 3$, then by the Sobolev embedding theorem, $B \in L^\infty((0, +\infty);\ C^2(R^2))$.

3. GLOBAL EXISTENCE

In this section, we prove that the solution to the Cauchy problem (1.1)–(1.2) of the matrix nonlinear Schrödinger equation is global under the condition

$$\|B_0\|_{L^2(R^2)} < \|\varphi\|_{L^2(R^2)}.$$

Let B be a local solution as given in Section 2, and let the maximal time interval of existence be $[0, T_{\max})$. We will show that $T_{\max} = +\infty$.

We need the following lemma:

LEMMA 3.1 [12]. *We have*

$$\|f\|_{L^{2\sigma+2}(R^N)}^{2\sigma+2} \le C_{\sigma,N}^{2\sigma+2} \|\nabla f\|_{L^2(R^N)}^{\sigma N} \cdot \|f\|_{L^2(R^N)}^{2+\sigma(2-N)}, \tag{3.1}$$

if $0 < \sigma < \frac{2}{N-2}$, $N \ge 2$. *Moreover, the best possible constant* $C_{\sigma,N}$ *for the above interpolation estimate is*

$$C_{\sigma,N} = \left(\frac{\sigma+1}{\|\varphi\|_{L^2(R^N)}^{2\sigma}} \right)^{\frac{1}{2\sigma+2}}. \tag{3.2}$$

where φ *is the ground state solution of*

$$\frac{\sigma N}{2} \Delta\varphi - \left(1 + \frac{\sigma}{2}(2-N)\right)\varphi + \varphi^{2\sigma+1} = 0. \tag{3.3}$$

Recall the two conservation laws for the L^2-norm and the Hamiltonian, namely,

$$\int_{R^2} |B|^2 dx = \int_{R^2} |B_0|^2 dx \tag{3.4}$$

and

$$H(B) = \int_{R^2} |\nabla B|^2 - |BB^*|^2 dx \triangleq H(B_0). \tag{3.5}$$

Proof of Theorem 1. *Step* 1. By the interpolation inequality (3.1), we have

$$\int_{R^2} |B|^4 dx \le C_{1,2}^4 \int_{R^2} |\nabla B|^2 dx \cdot \int_{R^2} |B|^2 dx. \tag{3.6}$$

It follows from (3.5) that, for all $t \in [0, T_{\max})$,

$$\begin{aligned} \int_{R^2} |\nabla B|^2 dx &= H(B_0) + \int_{R^2} |BB^*|^2 dx \\ &\le H(B_0) + C_{1,2}^4 \int_{R^2} |\nabla B|^2 dx \int_{R^2} |B_0|^2 dx. \end{aligned}$$

Thus,

$$\left(1 - C_{1,2}^4 \int_{R^2} |B_0|^2 dx\right) \int_{R^2} |\nabla B|^2 dx \le H(B_0),$$

$$\left(1 - \frac{\|B_0\|_{L^2(R^2)}^2}{\|\varphi\|_{L^2(R^2)}^2}\right) \int_{R^2} |\nabla B|^2 dx \le H(B_0).$$

So,

$$\int_{R^2} |\nabla B|^2 dx \le C \qquad \text{for all } t \in [0, T_{\max}). \tag{3.7}$$

Combining (3.4) and (3.7), and using the Sobolev imbedding theorem, we have

$$\|B\|_{L^p(R^2)} \le C, \qquad \text{for all } p > 2,\ t \in [0, T_{\max}). \tag{3.8}$$

It follows from Strichartz' estimate (see, e.g., Cazenave [4, Theorem 3.2.5]) that

$$\|B\|_{L^\infty((0,t), W^{1,r}(R^2))} \le C\|B_0\|_{H^1(R^2)} + C\|BB^*B\|_{L^1((0,t), W^{1,r'}(R^2))} \tag{3.9}$$

for all $t \in (0, T_{\max})$, $r > 2$, $1/r + 1/r' = 1$, where C is independent of t.

On the other hand, it follows from Hölder's inequality, (3.7) and (3.8), that

$$\begin{aligned} &\|BB^*B\|_{L^1((0,t), W^{1,r'}(R^2))} \\ &\quad \le C\int_0^t \|B\|_{L^{3r'}(R^2)}^3 dt + C\int_0^t \left(\int_{R^2} |B|^{r'(2-r')} dx\right)^{(1-\frac{r'}{2})} dt \cdot \left(\int_{R^2} |\nabla B|^2 dx\right)^{\frac{r'}{2}} \\ &\quad \le Ct, \end{aligned} \tag{3.10}$$

where C depends only on B_0.

Thus,

$$\|B\|_{W^{1,r}(R^2)} \le C\|B_0\|_{H^1(R^2)} + Ct, \qquad \text{for all } t \in [0, T_{\max}). \tag{3.11}$$

By the Sobolev imbedding theorem, one gets

$$\|B\|_{L^\infty(R^2)} \le C(t), \qquad \text{for all } t \in [0, T_{\max}), \tag{3.12}$$

where C depends only on t and B_0.

As in the proof of Lemma 2.2, we have

$$\frac{d}{dt}\int_{R^2} |\nabla^2 B|^2 dx \le C\int_{R^2} |\nabla^2 B|^2 |B|^2 + |\nabla^2 B|\,|\nabla B|^2 |B| dx. \tag{3.13}$$

By (3.11) and the Sobolev imbedding theorem, we have

$$\int_{R^2} |\nabla^2 B|^2 |B|^2 dx \le \|B\|_{L^\infty}^2 \int_{R^2} |\nabla^2 B|^2 dx$$

$$\le C(1+t) \int_{R^2} |\nabla^2 B|^2 dx, \tag{3.14}$$

for all $t \in (0, T_{\max})$,

$$\int_{R^2} |\nabla^2 B|\,|\nabla B|^2 |B| dx \le \|B\|_{L^\infty} \left(\int_{R^2} |\nabla^2 B|^2 dx \right)^{\frac{1}{2}} \left(\int_{R^2} |\nabla B|^4 dx \right)^{\frac{1}{2}}$$

$$\le C(1+t) \left(\int_{R^2} |\nabla^2 B|^2 dx \right)^{\frac{1}{2}}. \tag{3.15}$$

Hence

$$\frac{d}{dt} \int_{R^2} |\nabla^2 B|^2 dx \le C(1+t) \int_{R^2} |\nabla^2 B|^2 + C(1+t)^2 dx.$$

By Gronwall's inequality, we have

$$\int_{R^2} |\nabla^2 B|^2 dx \le C(t, B_0). \tag{3.16}$$

By Strichartz' estimate for the equations satisfied by ∇B, one gets, for any $r > 2$,

$$\|B\|_{W^{2,r}(R^2)} \le C(t, B_0), \qquad t \in [0, T_{\max}). \tag{3.17}$$

So, using the Sobolev imbedding theorem as above, we have

$$\frac{d}{dt} \int_{R^2} |\nabla^3 B|^2 dx \le C \int_{R^2} |\nabla^3 B|^2 |B|^2 + |\nabla^3 B|\,|\nabla^2 B|\,|\nabla B|\,|B| + |\nabla^3 B|\,|\nabla B|^3 dx$$

$$\le C_1(t) \int_{R^2} |\nabla^3 B|^2 dx + C_2(t), \qquad t \in (0, T_{\max}), \tag{3.18}$$

where $C_1(t), C_2(t)$ depends only on t and B_0.

By Gronwall's inequality, we obtain

$$\int_{R^2} |\nabla^3 B|^2 dx \le C(t), \qquad t \in (0, T_{\max}), \tag{3.19}$$

where C depends only on t and B_0.

In fact, we have by induction that, for all $t \in [0, T_{\max})$,

$$\|B\|_{H^l(R^2)} \le C(t, B_0), \tag{3.20}$$

where $C(t)$ depends only on t and B_0.

Step 2. Now suppose that $T_{\max} < +\infty$. Then for $0 < \delta < T_{\max}$, we can solve the Cauchy problem (1.1)–(1.2) for a B_1 which satisfies the Cauchy data

$$B_1(x, T_{\max} - \delta) = B(x, T_{\max} - \delta).$$

By the local existence result of Section 2, such a solution B_1 exists on a time interval $[T_{\max} - \delta, T_{\max} - \delta + \eta)$ for some $\eta > 0$. Since we have uniform bounds (independent of δ) on $\|B\|_{H^l(R^2)}$ if $T_{\max} < \infty$ as given in (3.20), by Lemma 2.2, it follows that η is independent of δ. Thus, if we choose δ sufficiently small, we have

$$T_{\max} - \delta + \eta > T_{\max}. \tag{3.21}$$

However, by construction, B_1 and B coincide on $R^2 \times [T_{\max} - \delta, T_{\max})$, and therefore B_1 extends B beyond the maximal time interval of existence. This shows that $T_{\max} = +\infty$. This also completes the proof of Theorem 1.

4. CONSTRUCTION OF BLOW-UP SOLUTIONS

LEMMA 4.1. *Given any $x_1 \in R^2$ and any finite time $T < +\infty$, one can construct a solution of the Cauchy problem* (1.1)–(1.2) *which blows up at* (x_1, T).

Proof. Without loss of generality, we let $x_1 = 0$. Let Q be a positive solution of the equation

$$\Delta u - u + u^3 = 0 \qquad \text{in } R^2 \tag{4.1}$$

of minimal L^2-norm (the ground state).

Define

$$\psi(x,t) = \frac{1}{(T-t)} \exp\left(-\frac{i|x|^2}{4(T-t)} + \frac{i}{T-t} \right) Q\left(\frac{x}{T-t} \right). \tag{4.2}$$

Then, $\psi(x,t)$ is a solution to the problem

$$\begin{cases} \psi_t = i\left(\Delta\psi + 2|\psi|^2\psi\right) & \text{in } R^2 \times (0,T), \\ \psi(x,0) = \dfrac{1}{T}\exp\left(-\dfrac{i|x|^2}{4T} + \dfrac{i}{T} \right) Q\left(\dfrac{x}{T} \right) & \text{in } R^2. \end{cases} \tag{4.3}$$

Define

$$B_1 = \begin{pmatrix} \psi & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$B_*(x) = B_1(x, 0).$$

Then B satisfies

$$\begin{cases} B_t = i(\Delta B + 2BB^*B), & \text{in } R^2 \times (0,T) \\ B(x,0) = B_*(x), & \text{in } R^2. \end{cases} \tag{4.4}$$

Moreover, it is easy to check that [3, 8, 9]

$$\|B_*\|_{L^2(R^2)} = \|Q\|_{L^2(R^2)}, \tag{4.5}$$

$$|B(x,t)|^2 \rightharpoonup \|Q\|^2_{L^2(R^2)} \delta_{x=0} \qquad \text{as } t \to T. \tag{4.6}$$

Thus, the solution blows up at $(0, T)$.

Remark. As consequence of Lemma 4.1, we have that the condition (1.3) is sharp.

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